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Use of Tangent and Normal Vectors for the Derivation of Equations of Tangent and Normal

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Abstract---*Remaining within the frame work of vector algebra and vector calculus, this paper makes use of the tangent and normal vectors to a given curve at a given point on it for the derivation of the equations of tangent and normal to the curve at that point. The techniques of derivation offered are generalized, simple and straight forward. Furthermore, unlike the traditional techniques, the present scheme increases the range of applicability of one of the fundament concepts of vector calculus (namely, gradient of a scalar point function) as well. As a result, this contribution must have educational value and it will enrich and sophisticate the traditional literature thereby enhancing the same as well.*
Keywords---*cartesian coordinate geometry, vector algebra, dot product, cross product, rectangular unit vectors, gradient of a scalar point function.*

Introduction

One of the important topics in the study of the traditional two dimensional Cartesian coordinate geometry (Todhunter, 2023; Baker, 1905), is "Tangent and Normal". Along with many other issues, the chapter on "Tangent and Normal" in traditional texts is concerned with the techniques of derivation of the equations of tangent and normal to a given curve at a given point. The curves considered in standard texts (Todhunter, 2023; Baker, 1905) are usually circle, parabola, ellipse, and hyperbola. Both the Calculus treatment (De, 1998; Nag, 1997) as well as the non-Calculus treatment (Todhunter, 2023; Baker, 1905) for the derivation of equations of tangent and normal are available.

Unlike the traditional schemes (Todhunter, 2023; Baker, 1905; De, 1998; Nag, 1997), the tangent and normal vectors to a given curve at a given point on it have been employed in this paper for the derivation of the equations of the tangent and normal to the curve at the said given point. The overall procedure is based on vector algebra and it makes simultaneous use of one of the fundamental concepts of vector calculus, namely, gradient of a scalar point function (Wang et al., 2018). The techniques of derivation offered are simple, straight forward and generalized ones on account of the fact that they are equally applicable for any given curve, irrespective of its nature. As a result, they will enrich the traditional literature thereby enhancing the same as well (Spiegel, 1974; Liu et al., 2016)).

Preliminaries

Dot product: Geometrically, the dot product of two vectors **A** and **B**, denoted by **A** · **B**, is defined as **A** · **B** = |A| |B| cos θ, where θ is the angle between **A** and **B** such that $0^0 \leq \theta \leq 180^0$.
Algebraically, if $\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}$, and $\mathbf{B} = B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k}$, where **i**, **j**, and **k** are rectangular unit vectors, then the dot product of **A** and **B**, denoted by **A** · **B**, is defined as $\mathbf{A} \cdot \mathbf{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$.

Cross product: Geometrically, the cross product of two vectors **A** and **B**, denoted by **A** × **B** is defined as a vector orthogonal to both **A** and **B** such that **A**, **B**, and **A** × **B** form a right-handed system, the magnitude of the vector **A** × **B** being |A| |B| sin θ, where θ is the angle between **A** and **B** such that $0^0 \leq \theta \leq 180^0$.

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Algebraically, if **A** = A_1 **i** + A_2 **j** + A_3 **k**, and **B** = B_1 **i** + B_2 **j** + B_3 **k**, where **i**, **j**, and **k** are rectangular unit vectors, then the cross product of **A** and **B**, denoted by **A** × **B**, is defined by the following determinant of third order.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Scalar point function: If corresponding to each and every point (x, y, z) in a certain region of space there exists a definite scalar φ(x, y, z), then φ(x, y, z) is called a scalar point function.

Gradient of a scalar point function: If the scalar point function $\varphi(x, y, z)$ is defined and differentiable at each and every point in a certain region of space, then gradient of φ(x, y, z), written in short as grad φ, is defined as, grad φ = $\mathbf{i}\,\frac{\partial\varphi}{\partial x} + \mathbf{j}\,\frac{\partial\varphi}{\partial y} + \mathbf{k}\,\frac{\partial\varphi}{\partial z}$, where **i**, **j**, and **k** are rectangular unit vectors (Candela-Munoz & Rodríguez-Gámez, 2023).
Now, let $\varphi(x, y, z) = k$, where k is a constant, be a level surface, i.e. a surface corresponding to each point of which the scalar point function φ has got the same constant value k. Let us consider any point P on this level surface defined by the position vector **r**.

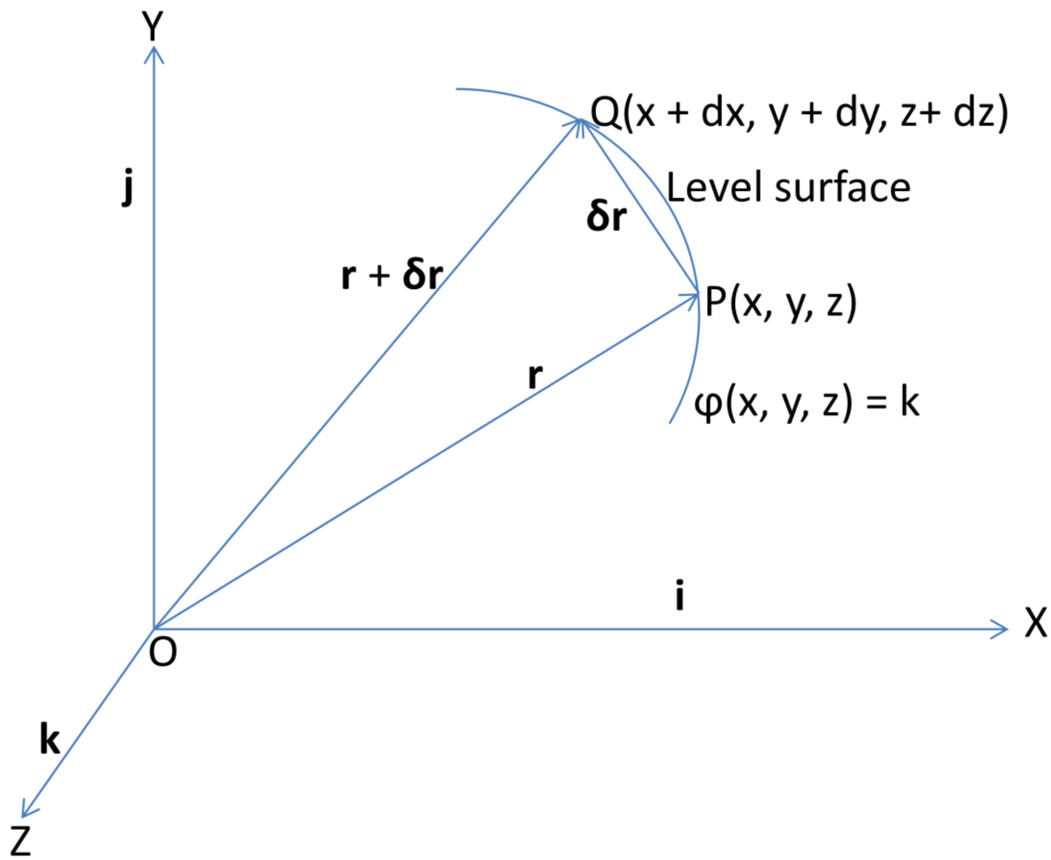

Figure 1. Diagram for finding the direction of gradient of a scalar point function

The position vector of a point Q very close to the point P and lying on the same level surface may then be taken as, $\mathbf{r} + \boldsymbol{\delta}\mathbf{r}$. If the Cartesian coordinates of the point P are (x, y, z), then the Cartesian coordinates of the point Q may be considered as $(x + dx, y + dy, z + dz)$. From Figure 1, we have, $\boldsymbol{\delta}\mathbf{r} = \mathbf{OQ} - \mathbf{OP}$. Now, we have, (grad φ) · $\boldsymbol{\delta}\mathbf{r} = \left(\mathbf{i}\,\frac{\partial\varphi}{\partial x} + \mathbf{j}\,\frac{\partial\varphi}{\partial y} + \mathbf{k}\,\frac{\partial\varphi}{\partial z}\right) \cdot (\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz) = \frac{\partial\varphi}{\partial x}\,dx + \frac{\partial\varphi}{\partial y}\,dy + \frac{\partial\varphi}{\partial z}\,dz = d\varphi$. Now, since both the points P and Q lie on the same level surface defined by $\varphi(x, y, z) = k$, the value of φ at each of the points P and Q must be each equal to k. Thus we must have, $d\varphi = 0$ in this case. It then follows from the aforesaid relation that, (grad φ) · **δr** = 0. Thus **grad φ** is perpendicular to **δr**. Now since the points P and Q are very close to each other, **δr** may be assumed to lie on the level surface under consideration (Mikhalev & Oseledets, 2018). Thus grad φ is a vector which is normal to the level surface $\varphi(x, y, z) = k$ at the point (x, y, z). It would be worth mentioning here that a similar theory on level surface could be developed in two-dimensional space, i.e. in the xy-plane.

Deriving Equation of Tangent

● **To find the equation of the tangent to the circle** $x^2 + y^2 = a^2$ **at the point** (x_1, y_1)
Here, the equation of the circle is $x^2 + y^2 = a^2$, i.e. $\varphi(x, y) = 0$, where $\varphi(x, y) = x^2 + y^2 - a^2$ is a scalar point function. Then considering Figure 2, we have, **QN** = (grad φ) at the point (x_1, y_1), where QN is the normal to the circle $x^2 + y^2 = a^2$ at the point $Q(x_1, y_1)$. Now, grad φ = $2x\,\mathbf{i} + 2y\,\mathbf{j}$. Hence **QN** = $2x_1\,\mathbf{i} + 2y_1\,\mathbf{j}$.

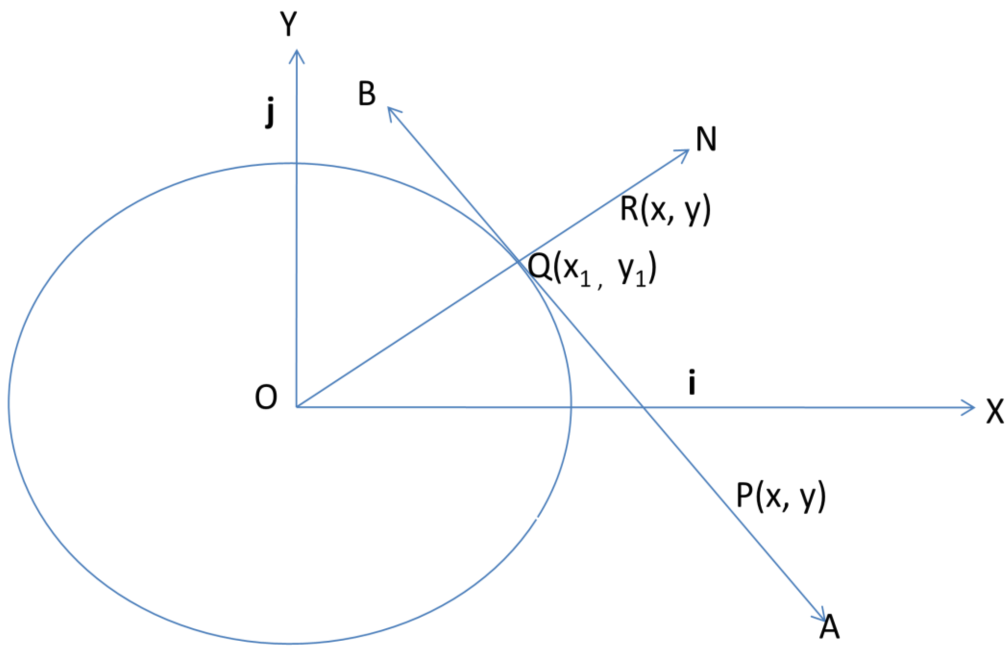

Figure 2. Diagram for finding the equations of the tangent and normal to the circle $x^2 + y^2 = a^2$ at the point (x_1, y_1)

As shown in Figure 2, let $P(x, y)$ be any point on the tangent. Then **QP** = $(x - x_1)\,\mathbf{i} + (y - y_1)\,\mathbf{j}$. Now, since **QN** is perpendicular to **QP**, we must have, **QN**· **QP** = 0. i.e. $(2x_1\,\mathbf{i} + 2y_1\,\mathbf{j}) \cdot \{(x - x_1)\,\mathbf{i} + (y - y_1)\,\mathbf{j}\} = 0$. Remembering that the point (x_1, y_1) lies on the circle $x^2 + y^2 = a^2$, this ultimately leads to the relation $xx_1 + yy_1 = a^2$, which is the required equation of the tangent.

● **To find the equation of the tangent to the circle** $x^2 + y^2 + 2gx + 2fy + c = 0$ **at the point** (x_1, y_1).
Here, $\varphi(x, y) = x^2 + y^2 + 2gx + 2fy + c$ is the scalar point function under consideration. Then considering Figure 3, we have, **QN** = (grad φ) at the point (x_1, y_1), where QN is the normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point $Q(x_1, y_1)$. Now, grad φ = $2(x + g)\,\mathbf{i} + 2(y + f)\,\mathbf{j}$. Hence **QN** = $2(x_1 + g)\,\mathbf{i} + 2(y_1 + f)\,\mathbf{j}$.

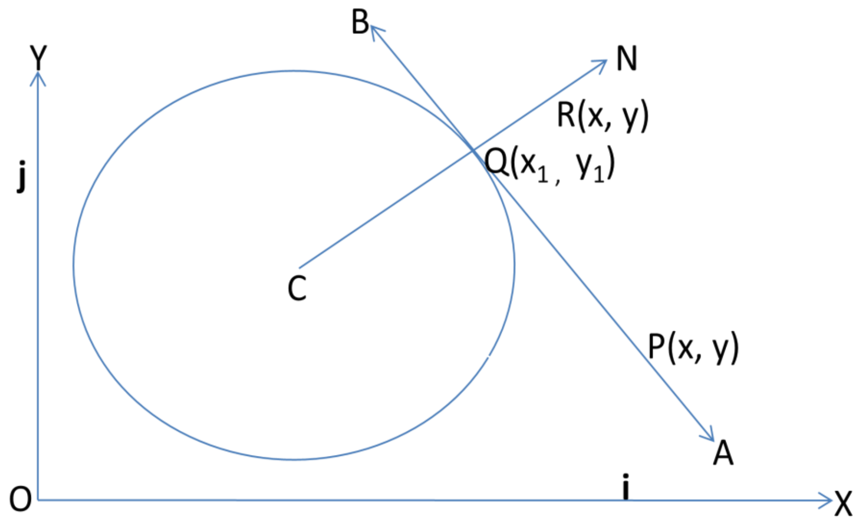

Figure 3. Diagram for finding the equations of the tangent and normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1)

As shown in Figure 3, let $P(x, y)$ be any point on the tangent. Then **QP** = $(x - x_1)$ **i** + $(y - y_1)$ **j**. Now, since **QN** is perpendicular to **QP**, we must have, **QN** · **QP** = 0. i.e. $\{2(x_1 + g)$ **i** + $2(y_1 + f)$ **j**$\}$ · $\{(x - x_1)$ **i** + $(y - y_1)$ **j**$\}$ = 0. Remembering that the point (x_1, y_1) lies on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, this ultimately leads to the relation $xx_1 + yy_1 + g(x + x_1) + f(y + y) + c = 0$, which is the required equation of the tangent.

● **To find the equation of the tangent to the parabola** $y^2 = 4ax$ **at the point** (x_1, y_1)

Here, $\varphi(x, y) = y^2 - 4ax$ is the scalar point function to be considered. Then considering Figure 4, we have, **QN** = (grad φ) at the point (x_1, y_1), where QN is the normal to the parabola $y^2 = 4ax$ at the point $Q(x_1, y_1)$. Now, grad φ = $-4a$ **i** + $2y$ **j**. Hence **QN** = $-4a$ **i** + $2y_1$ **j**. As shown in Figure 4, let $P(x, y)$ be any point on the tangent. Then **QP** = $(x - x_1)$ **i** + $(y - y_1)$ **j**. Now, since **QN** is perpendicular to **QP**, we must have, **QN** · **QP** = 0. i.e. $(-4a$ **i** + $2y_1$ **j**$)$ · $\{(x - x_1)$ **i** + $(y - y_1)$ **j**$\}$ = 0. Remembering that the point (x_1, y_1) lies on the parabola $y^2 = 4ax$, this ultimately leads to the relation $yy_1 = 2a(x + x_1)$, which is the required equation of the tangent (Yamanaka et al., 2008).

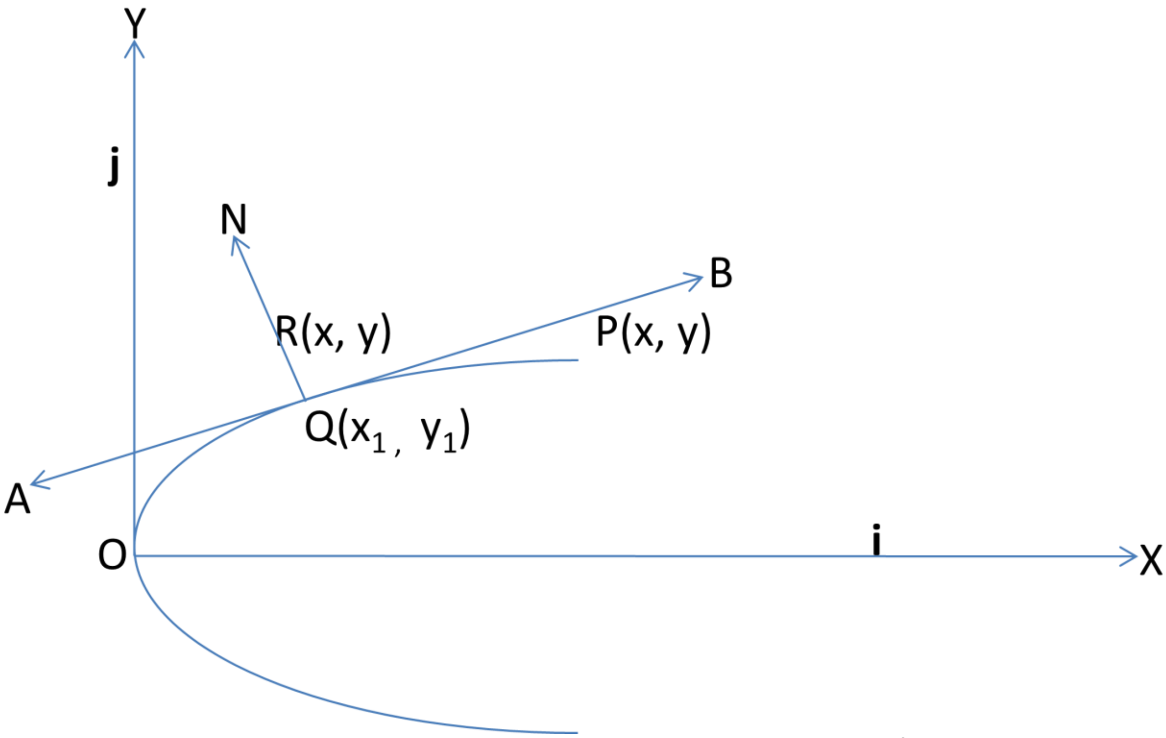

Figure 4. Diagram for finding the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(x_{1\,,}\,y_1)$

● **To find the equation of the tangent to the ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **at the point** $(x_{1\,,}\,y_1)$

Here, $\varphi(x_{,}\,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ is the scalar point function under consideration. Then considering Figure 5, we have, **QN** = (grad φ) at the point $(x_{1\,,}\,y_1)$, where QN is the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $Q(x_{1\,,}\,y_1)$. Now, grad $\varphi = \left(\frac{2x}{a^2}\right)$ **i** $+ \left(\frac{2y}{b^2}\right)$ **j**. Hence, **QN** $= \left(\frac{2x_1}{a^2}\right)$ **i** $+ \left(\frac{2y_1}{b^2}\right)$ **j**.

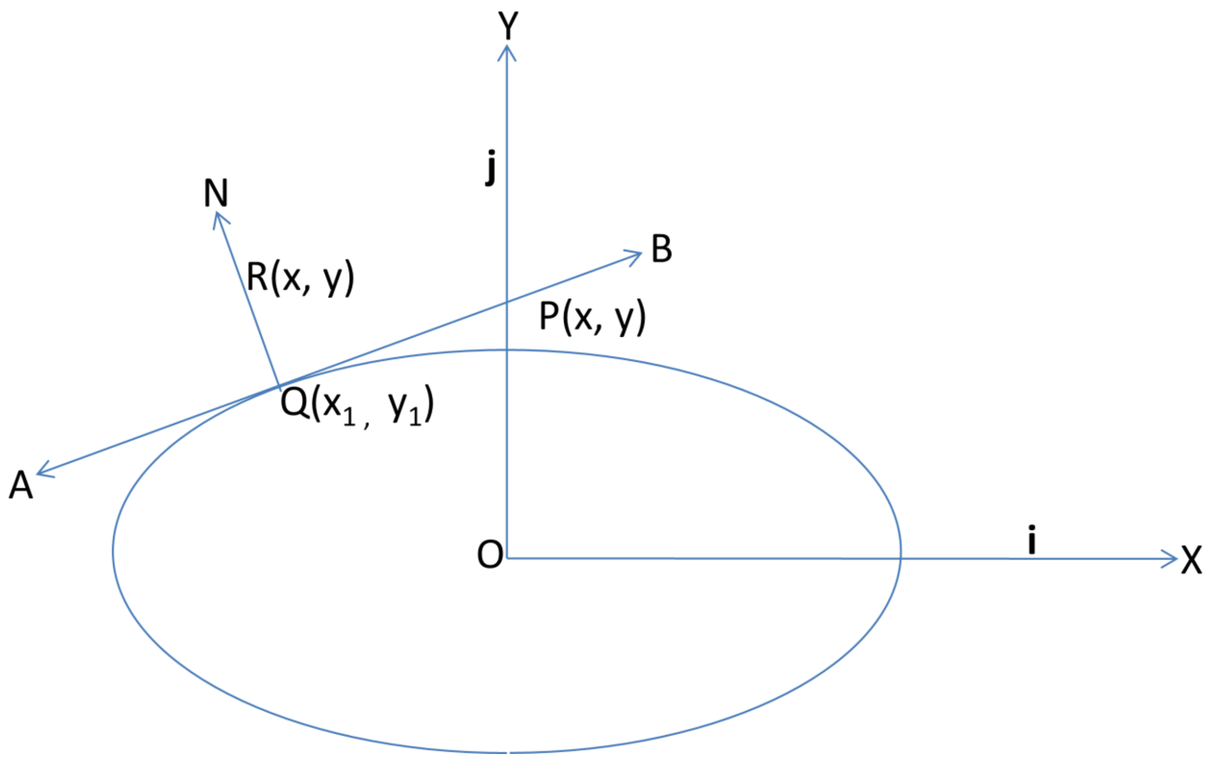

Figure 5. Diagram for finding the equations of the tangent and normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1)

As shown in Figure 5, let $P(x, y)$ be any point on the tangent. Then **QP** = $(x - x_1)$ **i** + $(y - y_1)$ **j**. Now, since **QN** is perpendicular to **QP**, we must have, **QN** · **QP** = 0. i.e. $\{\left(\frac{2x_1}{a^2}\right)\mathbf{i} + \left(\frac{2y_1}{b^2}\right)\mathbf{j}\} \cdot \{(x - x_1)\ \mathbf{i} + (y - y_1)\ \mathbf{j}\} = 0$. Remembering that the point (x_1, y_1) lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, this ultimately leads to the relation $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$, which is the required equation of the tangent.

● **To find the equation of the tangent to the hyperbola** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ **at the point** (x_1, y_1) Here, $\varphi(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$ is the scalar point function to be considered. Then considering Figure 6, we have, **QN** = (grad φ) at the point (x_1, y_1), where QN is the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $Q(x_1, y_1)$.

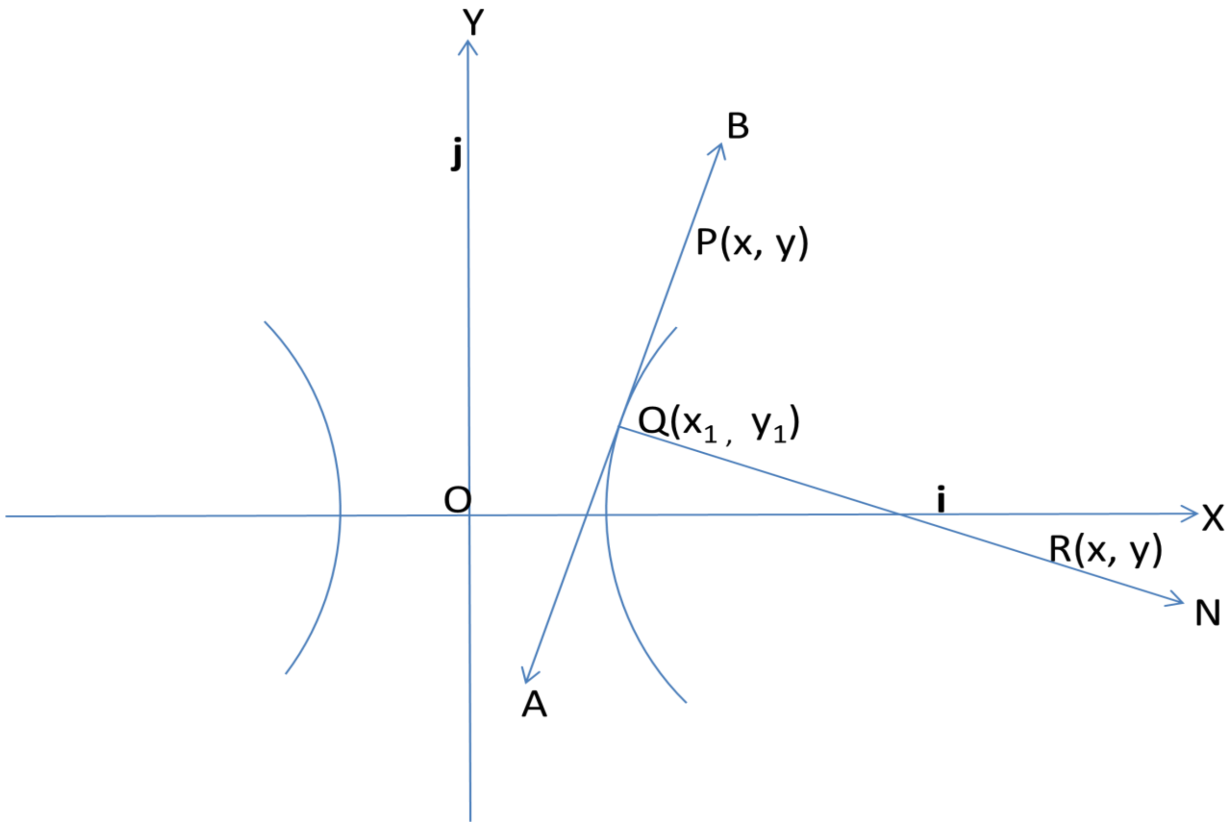

Figure 6. Diagram for finding the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(x_{1,} y_1)$

Now, grad φ = $\left(\frac{2x}{a^2}\right)$ **i** $-\left(\frac{2y}{b^2}\right)$ **j**. Hence **QN** = $\left(\frac{2x_1}{a^2}\right)$ **i** $-$ $\left(\frac{2y_1}{b^2}\right)$ **j**.
As shown in Figure 6, let $P(x_{,} y)$ be any point on the tangent. Then **QP** = $(x - x_1)$ **i** + $(y - y_1)$ **j**. Now, since **QN** is perpendicular to **QP**, we must have, **QN** · **QP** = 0. i.e. $\{\left(\frac{2x_1}{a^2}\right)$ **i** $-\left(\frac{2y_1}{b^2}\right)$ **j**$\}$ · $\{(x - x_1)$ **i** + $(y - y_1)$ **j**$\}$ = 0. Remembering that the point $(x_{1,} y_1)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, this ultimately leads to the relation $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$, which is the required equation of the tangent.

Deriving Equation of Normal

● **To find the equation of the normal to the circle** $x^2 + y^2 = a^2$ **at the point** $(x_{1,} y_1)$
Let us consider Figure 2 again. Now, let $R(x, y)$ be any point on the normal QN. Then we have, **QR** = $(x - x_1)$ **i** + $(y - y_1)$ **j**. Further as has been found earlier, in this case we have, **QN** = $2x_1$ **i** + $2y_1$ **j**. Now, since **QR** and **QN** are collinear vectors, we must have, **QR** × **QN** = **0**. i.e. $\{(x - x_1)$ **i** + $(y - y_1)$ **j**$\}$ × $(2x_1$ **i** + $2y_1$ **j** $)$ = **0**, which finally leads to the equation $xy_1 - yx_1 = 0$, which is the required equation of the normal.

● **To find the equation of the normal to the circle** $x^2 + y^2 + 2gx + 2fy + c = 0$ **at the point** $(x_{1,} y_1)$
Considering Figure 3 again let $R(x, y)$ be any point on the normal QN. Then we have, **QR** = $(x - x_1)$ **i** + $(y - y_1)$ **j**. Further as has been found earlier, in this case we have, **QN** = $2(x_1 + g)$ **i** + $2(y_1 + f)$ **j**. Now, since **QR** and **QN** are collinear vectors, we must have, **QR** × **QN** = **0**. i.e. $\{(x - x_1)$ **i** + $(y - y_1)$ **j**$\}$ × $\{2(x_1 + g)$ **i** + $2(y_1 + f)$ **j**$\}$ = **0**, which finally leads to the equation $(y_1 + f)x - (x_1 + g)y - fx_1 + gy_1 = 0$, which is the required equation of the normal (Moon & Kota, 2002).

● **To find the equation of the normal to the parabola** $y^2 = 4ax$ **at the point** (x_1, y_1)

Considering Figure 4 again, let $R(x, y)$ be any point on the normal QN. Then we have, **QR** = $(x - x_1)$ **i** + $(y - y_1)$ **j**. Also, in this case we have, **QN** = $-4a$ **i** + $2y_1$ **j**. Now, since **QR** and **QN** are collinear vectors, we must have, **QR** × **QN** = **0**, i.e. $\{(x - x_1)\, \mathbf{i} + (y - y_1)\, \mathbf{j}\} \times \{-4a\, \mathbf{i} + 2y_1\, \mathbf{j}\} = \mathbf{0}$, which ultimately leads to the relation $y - y_1 = -\frac{y_1}{2a}(x - x_1)$, which is the required equation of the normal.

● **To find the equation of the normal to the ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **at the point** (x_1, y_1)

Considering Figure 5 again, let $R(x, y)$ be any point on the normal QN. Then we have, **QR** = $(x - x_1)$ **i** + $(y - y_1)$ **j**. Also, in this case we have, **QN** = $\left(\frac{2x_1}{a^2}\right)$ **i** + $\left(\frac{2y_1}{b^2}\right)$ **j**. Now, since **QR** and **QN** are collinear vectors, we must have, **QR** × **QN** = **0**. i.e. $\{(x - x_1)\, \mathbf{i} + (y - y_1)\, \mathbf{j}\} \times \left\{\left(\frac{2x_1}{a^2}\right) \mathbf{i} + \left(\frac{2y_1}{b^2}\right) \mathbf{j}\right\} = \mathbf{0}$, which ultimately yields the relation $y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$, which is the required equation of the normal.

● **To find the equation of the normal to the hyperbola** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ **at the point** (x_1, y_1)

Considering Figure 6 again, let $R(x, y)$ be any point on the normal QN. Then we have, **QR** = $(x - x_1)$ **i** + $(y - y_1)$ **j**. Also, in this case we have, **QN** = $\left(\frac{2x_1}{a^2}\right)$ **i** $-$ $\left(\frac{2y_1}{b^2}\right)$ **j**. Now, since **QR** and **QN** are collinear vectors, we must have, **QR** × **QN** = **0**. i.e. $\{(x - x_1)\, \mathbf{i} + (y - y_1)\, \mathbf{j}\} \times \left\{\left(\frac{2x_1}{a^2}\right) \mathbf{i} - \left(\frac{2y_1}{b^2}\right) \mathbf{j}\right\} = \mathbf{0}$, which ultimately yields the relation $y - y_1 = -\frac{a^2 y_1}{b^2 x_1}(x - x_1)$, which is the required equation of the normal.

Conclusion

Remaining within the framework of vector algebra and vector calculus, this paper unlike the long-running techniques (Todhunter, 2023; Baker, 1905; De, 1998; Nag, 1997), demonstrates how the tangent and normal vectors to a given curve at a given point on it could be employed to derive the equations of the tangent and normal to the curve at the said given point. Strong points of the paper are the following:

1) Use of the tangent and normal vectors for the derivation of the equations of the tangent and normal exists nowhere in the traditional literature. To that extent, the present scheme is being claimed to be novel.
2) This work, based totally on Vector algebra and one of the fundamental concepts of Vector calculus, namely "Gradient of a scalar point function", enhances the current literature by proposing a novel, generalized, simple, and straight-forward technique for the derivation of the equations of tangent and normal to a given curve at a given point on it by making use of the tangent and normal vectors to the said curve at the said given point.
3) Incorporation of the application of the tangent and normal vectors to a given curve at a given point on it in the technique of derivation of the equations of the tangent and normal is being claimed to be the added value of the work presented.
4) The present scheme increases the range of applicability of vector algebra and vector calculus
5) The treatments of derivation offered are generalized, simple and straight forward.
6) The present scheme has got educational value and it will enrich the relevant literature.
7) It will enhance deepening of thought and understanding regarding vector calculus and its applications as well.

On account of lack of facilities available, the scheme offered in this paper has not yet been practically tested. Field trials would be necessary to make definite comments regarding the actual performance of the present scheme over those of the existing ones.

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