



Reducing the PDEs to ODEs Through Lie Vectors Using the Integrated Factors



Tony Brian Fogel^a
Steven Segan Lucky^b
James Arnold Mandy^c
Berth Bryan Wellbirth^d

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Abstract

We reduce the PDEs to ODEs through Lie vectors as previously done through two reduction stages. Some of these ODEs have no solution. Some researchers in this step, use the SMM, power series method or Riccati equation method to solve non-solvable equations. We use the integrating factors as a tool to reduce the order and the nonlinearity in an ODE. This explores new solutions as it appears for the (2+1)-dimensional (CBS) and (3+1)-dimensional generalized BKP solutions compared results.

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Author correspondence:

Tony Brian Fogel,
Cornell University, Private university in Ithaca, NY 14850, USA.
Email address: fogel.tb@cornell.edu

1. Introduction

Recently, many methods are applied to reach analytical solutions of NLPDEs as Darboux transformation (Tang & Xie, 2011). Moghadam *et al.*, (2013), the (G'/G) expansion method, Exp-Function method and Rational Function transformations (Budd *et al.*, 2006). Here, two equations namely, the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation and the (3+1)-dimensional generalized BKP equation have been solved by Lie symmetry Reduction method (Ahmad *et al.*, 2015). Using the Lie Reduction method and (IF), we derive a novel combination of solutions for these equations.

^a Cornell University, Ithaca, US

^b Cornell University, Ithaca, US

^c Columbia University, New York, US

^d Columbia University, New York, US

2. Materials and Methods

Mathematical Formulation

In this paragraph, we reduce the nonlinear evolution equations to (ODEs) in three steps. For each used Lie vector, we apply the following steps;

- The independent variables $(x; y; z; t)$ are reduced to a (PDE) in two variables (r, s) .
- Evaluate Lie infinitesimals for these PDE then use the evaluated symmetries for a reduction of independent variables from $(r; s)$ to one variable (η) .
- The reduced ODE is non-solvable equations, through their corresponding IF are reduced to new solvable ones.

(2+1)-Dimensional Calogero-Bogoyavlenskii-Schiff equation

$$u_{xt} + 4u_x u_{yx} + 2u_y u_{xx} + u_{xxx} = 0 \quad (1)$$

Where $u(x, y, t)$ describes the interaction of Riemann wave propagation in the x -direction with the long-wave propagation in the y -direction. [Tang & Xie, \(2011\)](#), derive analytical solutions of (1) using the Tanh-Coth method. [Wang et al., \(2015\)](#), obtained the Noether-type operators associated with the partial Lagrangian for all possible arbitrary functions, followed by a double reduction using symmetries. [Li & Wang \(2007\)](#), used Lie symmetry with the SMM to present new solutions of CBS equation (1).

Here, we apply two stages of symmetry reduction method to reduce the (CBS) equation to ODEs. During the reduction process, some of the obtained ODEs had no quadrature. We thus solve them using their (IF). Equation (1) has 24 Lie vectors. From the adjoint table, three optimal vectors are deduced.

$$\begin{aligned} X_2 &= \frac{\partial}{\partial t} + \frac{\partial}{\partial u}, & X_7 &= t \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + 0.5y \frac{\partial}{\partial u}, \\ X_{21} &= \left(1 + \frac{1}{4}tx\right) \frac{\partial}{\partial x} + \left(\frac{1}{2}ty\right) \frac{\partial}{\partial y} + \left(\frac{1}{2}t^2\right) \frac{\partial}{\partial t} + \left(-\frac{1}{4}tu + \frac{1}{8}xy\right) \frac{\partial}{\partial u} \end{aligned} \quad (2)$$

These vectors are utilized to reduce (CBS) (1) three independent variables; $(x; y; t)$ to two independent variables; $(r; s)$

Reduction of the independent variables in (CBS) equation using X_2 Lie vector

Equation (1) is transformed through the vector $X_2 = \frac{\partial}{\partial t} + \frac{\partial}{\partial u}$ to;

$$4 F_r F_{rs} + 2 F_s F_{rr} + F_{rrs} = 0 \quad (3)$$

Where $r = x$, $s = y$ and $F(r, s) = u(x, y, t) - t$. The above equation has no exact solution but has six Lie vectors. We choose here to deal only with V_1, V_5 Lie vector as they lead to ODE's having no closed-form solution.

$$V_1 = \frac{\partial}{\partial r} + \frac{\partial}{\partial s}, \quad V_5 = r \frac{\partial}{\partial r} + \frac{\partial}{\partial s} - F \frac{\partial}{\partial F}. \quad (4)$$

Reduction using V_1

Using V_1 (CBS) transform to a nonlinear fourth-degree ODE of the following form;

$$6\theta_{\eta\eta}\theta_\eta - \theta_{\eta\eta\eta\eta} = 0 \quad (5)$$

Using Integrating Factor to obtain an exact solution

We first deduce equation (5) IF using maple.

$$\mu_1 = \theta_\eta, \mu_2 = 1 \quad (6)$$

The IF reduce equation (5) to;

$$2\theta_\eta^3 - \theta_{\eta\eta}^2 = 0 \quad (7)$$

This equation has a closed-form solution of the form;

$$\theta(\eta) = -\frac{2}{\eta+c_1} + c_2 \quad (8)$$

Where $\eta = -r + s$, $\theta(\eta) = F(r, s)$ and c_1, c_2 are the integration constants. Back substituting to (x, y, t) for $r = x, s = y, F(r, s) = u(x, y, t) - t$, we get;

$$u(x, y, t) = -\frac{2}{-x + y + c_1} + c_2 + t \quad (9)$$

This solution in (9) is plotted in Fig. 1 for two different times; $t=0, t=20$ sec.

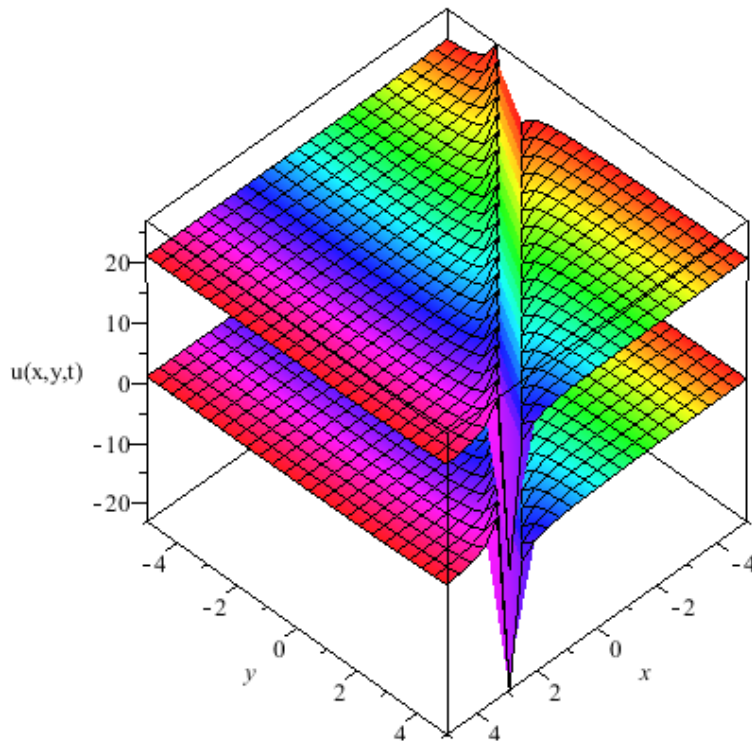


Figure 1. $u(x, y, t)$ at $c_1 = -2, c_2 = 1, t = 0, 20$ sec.

The wave in Fig. 1 shows a row of peakons moving downward as time passes from zero to twenty.

Reduction using V_5

Using the V_5 , (CBS) equation is reduced to a nonlinear ODE of the form;

$$10\theta_\eta^2 + 6\theta_\eta\theta_{\eta\eta} + 8\theta\theta_\eta + 4\theta\theta_{\eta\eta} - 6\theta_\eta - 11\theta_{\eta\eta} - 6\theta_{\eta\eta\eta} - \theta_{\eta\eta\eta\eta} = 0 \quad (10)$$

The above equation has no exact solution. Using the integrating factor, we can reduce it to a simpler equation.

We first investigate the IF of (10) using maple.

$$\mu_1 = e^{2\eta}, \mu_2 = e^{4\eta}\theta_\eta \tag{11}$$

The IF reduce equation (10) to;

$$2\theta_\eta^3 + 4\theta\theta_{\eta\eta}^2 - 3\theta_\eta^2 - 4\theta_\eta\theta_{\eta\eta} - \theta_{\eta\eta}^2 = 0 \tag{12}$$

This equation has an exact solution of the form;

$$\theta(\eta) = 2 + c_1 - \tanh(0.5\eta\sqrt{5 + 4c_1} + 0.5c_2\sqrt{5 + 4c_1})\sqrt{5 + 4c_1} \tag{13}$$

Where $\eta = -\ln(r) + s$, $\theta(\eta) = rF(r, s)$ and c_1, c_2 are integration constants.

Then back substituting to (x, y, t) where $r = x, s = y, F(r, s) = u(x, y, t) - t$, we obtain;

$$u(x, y, t) = \frac{2 + c_1 - \tanh(0.5(-\ln(x) + y)\sqrt{5 + 4c_1} + 0.5c_2\sqrt{5 + 4c_1})\sqrt{5 + 4c_1} + t}{x} \tag{14}$$

This solution is plotted in Fig. 2 in a complex domain. The peakon waves depicted in Fig. 2 decay with time.

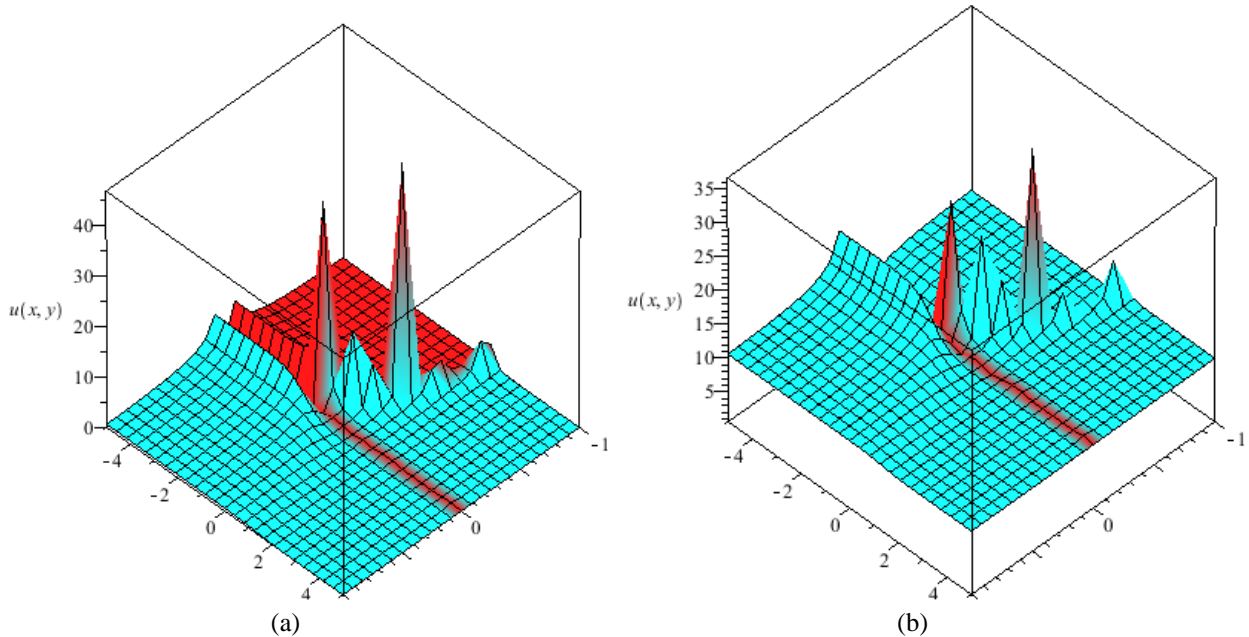


Figure 2. (a) $u(x, y, t)$ at $c_1 = 1, c_2 = 1, t = 0$ sec;

(b) $u(x, y, t)$ at $c_1 = 1, c_2 = 1, t = 10$ sec.

By comparison of the results obtained, our solution is new

Reduction of the independent variables in (CBS) equation using X_7 Lie vector

Equation (1) is transformed through the optimal vector X_7 to

$$16 F_s F_{rs} + 8 F_r (F_{ss}) - 8 F_{ssr} = 0 \tag{15}$$

Where $r = y, s = -2x + t^2, F(r, s) = u(x, y, t) - 0.5ty$,

This equation hasn't an exact solution but has a six Lie vectors. We choose to work only with V_1 . This Lie vector leads to an ODE with no analytic solution. While the rest of the vector lead to solvable ODEs.

$$V_1 = \frac{\partial}{\partial r} + \frac{\partial}{\partial s} \tag{16}$$

Using V_1 transform (CBS) to a nonlinear fourth-degree ODE of the following form;

$$3\theta_{\eta\eta}\theta_\eta - \theta_{\eta\eta\eta} = 0 \tag{17}$$

Using Integrating Factor to obtain a closed-form solution

We first investigate the IF of (17) using maple.

$$\mu_1 = \theta_\eta, \mu_2 = 1 \tag{18}$$

We use the two-integrating factor to reduce equation (17) to;

$$\theta_\eta^3 - \theta_{\eta\eta}^2 = 0 \tag{19}$$

This equation has an explicit solution in the form;

$$\theta(\eta) = -\frac{4}{\eta + c_1} + c_2 \tag{20}$$

where $\eta = -r + s$, $\theta(\eta) = F(r, s)$ and c_1, c_2 are integration constants.

Then back to (x, y, t) coordinates where $r = y, s = -2x + t^2, F(r, s) = u(x, y, t) - 0.5ty$, we obtain;

$$u(x, y, t) = -\frac{4}{-y - 2x + t^2 + c_1} + c_2 + 0.5ty \tag{21}$$

This solution is plotted in Fig. 3. The wave peak position changes with time.

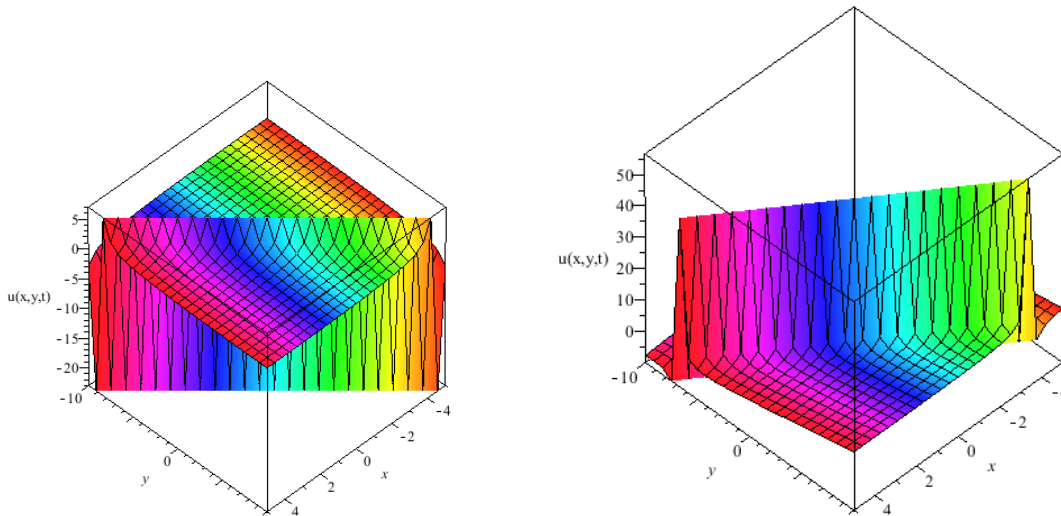


Figure 3. $u(x, y, t)$ at $c_1 = 1, c_2 = 1, t = 0, 1.5$ sec.

Reduction of the independent variables in (CBS) equation using X_{21} Lie vector

Equation (1) is transformed through the vector X_{21} to;

$$8F_r^2 + 2FF_{ss} + 4F_{sss} + 2r F_s F_{rs} + 4r F_r (F_{ss}) + s F_{ssss} = 0 \tag{22}$$

The reduced equation (22) has no closed-form solution but have six Lie vectors. We choose to work with V_2 , as follow;

$$V_2 = r \frac{\partial}{\partial r} + \left(\frac{1}{2}s + \sqrt{r}\right) \frac{\partial}{\partial s} - \frac{1}{2}F \frac{\partial}{\partial F} \tag{23}$$

Using V_2 transform (CBS) to a nonlinear fourth degree ODE in the following form;

$$6\theta_{\eta\eta}\theta_{\eta} + \theta_{\eta\eta\eta\eta} = 0 \tag{24}$$

Using Integrating Factor to obtain an exact solution

We get IF for (24) using maple.

$$\mu_1 = \theta_{\eta}, \mu_2 = 1 \tag{25}$$

The IF reduce equation (24) to;

$$2\theta_{\eta}^3 + \theta_{\eta\eta}^2 = 0 \tag{26}$$

This equation has a closed-form solution;

$$\theta(\eta) = \frac{2}{\eta + c_1} + c_2 \tag{27}$$

where $\eta = \frac{s-\sqrt{r}\ln(r)}{\sqrt{r}}$, $\theta(\eta) = \sqrt{r}F(r, s)$ and c_1, c_2 are integration constants.

Then back substituting to (x, y, t) where $r = \frac{t}{y}, s = \frac{3tx+4}{3\sqrt{y}t}, F(r, s) = -(-u(x, y, t) + \frac{yx}{4t} + \frac{y}{t^2})$ we obtain;

$$u(x, y, t) = \frac{2}{\frac{3tx+4}{3t} - \sqrt{t}\ln\left(\frac{t}{y}\right) + c_1} + c_2 \left[\frac{3tx+4}{3\sqrt{y}t} - \sqrt{\frac{t}{y}}\ln\left(\frac{t}{y}\right) + c_1 \right] + \frac{y}{t^2} + \frac{yx}{4t} \tag{28}$$

This solution is plotted in Fig. 4.

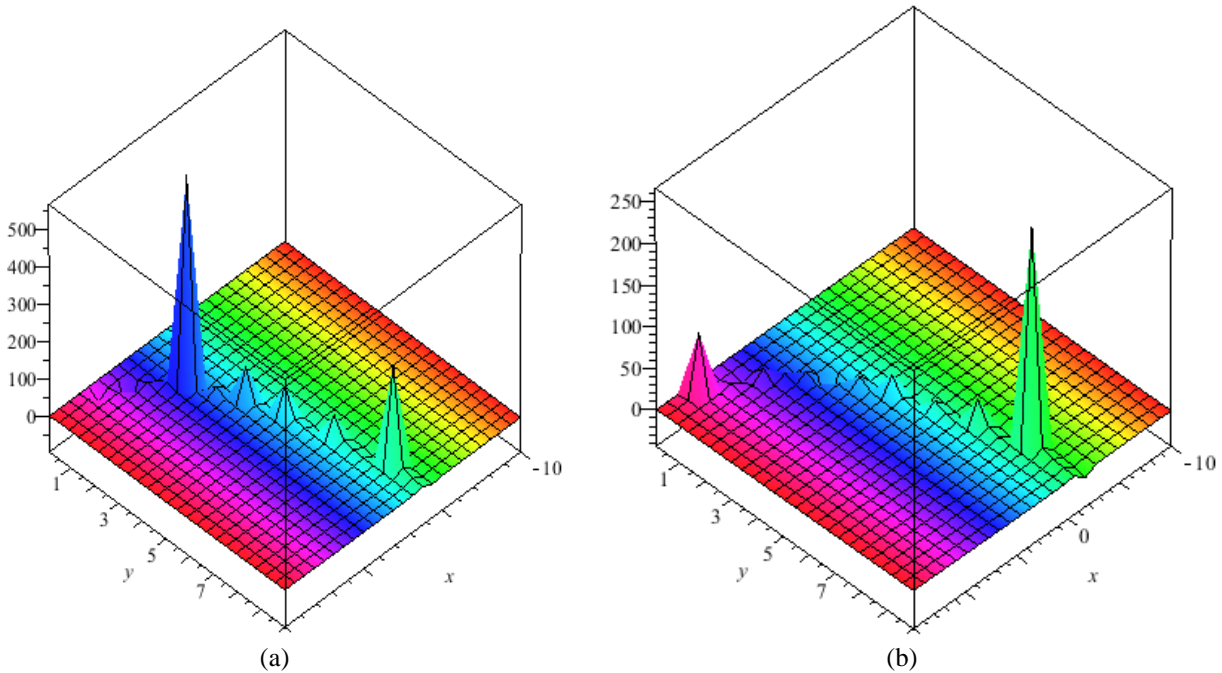


Figure 4. (a) $u(x, y, t)$ at $c_1 = 1, c_2 = 1, t = 5$ sec.

(b) $u(x, y, t)$ at $c_1 = 1, c_2 = 1, t = 20$ sec.

The peakon waves move on a parabolic axis drifting to the right with time. This result is new.

3. Results and Discussions

(3+1)-Dimensional B-Kadomtsev–Petviashvili (BKP) Equation

Susto & Krstic (2010), use Lie symmetry analysis, Riccati equation and power series method to introduce specific solutions for (3+1) BKP equation. Also, Hasan *et al.*, (2016), apply prorogation theorem to get the similarity variables and use these generators to solve the equation. In this paper, we consider the BKP equation in (3+1) dimensional space as;

$$u_{yt} - u_{xxxxy} - 3u_y u_{xx} - 3u_x u_{xy} + 6u_{xx} - 3u_{zz} = 0 \quad (29)$$

This equation has 36 Lie vectors, and we will choose X_{13} to reduce the equation;

$$X_{13} = \frac{\partial}{\partial z} + tz \frac{\partial}{\partial u}, X_1 + X_9 + X_{13} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + tz \frac{\partial}{\partial u} \quad (30)$$

The PDE (29) function of four independent variables; ($x; y; t; z$) is first reduced to a PDE in three independent variables, ($l; h; o$), using its Lie vectors (30) then reduce to two independent variables ($r; s$) and then one independent η .

Using X_{13} Lie Vector to reduce the Independent Variables in (BKP) Equation

Equation (29) is transformed through the vector $X_{13} = \frac{\partial}{\partial z} + tz \frac{\partial}{\partial u}$ to;

$$K_{oh} - (K_{llh}) - 3K_{ll}K_h - 3K_{lh}K_l + 6K_{ll} - 3o = 0 \quad (31)$$

This equation has no closed form solution, but possesses 12 Lie vectors; we choose V_1 ;

$$V_1 = \frac{\partial}{\partial l} + \frac{\partial}{\partial h} \quad (32)$$

This vector transforms the equation (31) to;

$$F_{rs} - 6F_r(F_{rr}) + 6F_{rr} + F_{rrrr} - 3s = 0 \quad (33)$$

This equation has no closed-form solution, but have eight Lie vectors; we will choose here to work only with e_5

$$e_5 = \left(\frac{1}{3}r - 8s^3\right) \frac{\partial}{\partial r} + s \frac{\partial}{\partial s} + \left(1 + 4rs^2 + \frac{2}{3}r - \frac{1}{3}F\right) \frac{\partial}{\partial F} \quad (34)$$

Using e_5 , transform (33) to the following form;

$$-54\theta_{\eta\eta} - \theta_{\eta\eta\eta\eta} + 18\theta_{\eta\eta}\theta_{\eta} = 0 \quad (35)$$

This is ODE has no analytical solution.

Using the Integrating Factor to obtain an explicit solution

We first deduce equation (35) IF using maple.

$$\mu_1 = \theta_{\eta}, \mu_2 = 1 \quad (36)$$

The IF reduce equation (35) to;

$$\theta_{\eta\eta}^2 - 6\theta_{\eta}^3 + 54\theta_{\eta}^2 = 0 \quad (17)$$

This equation has a closed-form solution of the form;

$$\theta(\eta) = \sqrt{6} \tan\left(\frac{3\sqrt{6}}{2}\eta + \frac{3\sqrt{6}}{2}c_1\right) - \sqrt{6} \tan^{-1}\left(\tan\left(\frac{3\sqrt{6}}{2}\eta + \frac{3\sqrt{6}}{2}c_1\right)\right) + 9\eta + c_2 \quad (38)$$

where $\eta = \frac{1}{3}r + s^3$, $\theta(\eta) = F(r, s) = /s + \frac{27^{\frac{2}{3}}}{5}s^5 - \frac{27^{\frac{2}{3}}}{6}s^2r - \frac{27^{\frac{2}{3}}}{2}s^5$.

Then back to (1; h; o) with $r = -l + h, s = o, F(r, s) = K(l, h, o)$; $h = y, l = x, o = t, K(l, h, o) = u(x, y, t, z) - 0.5z^2$ we obtain;

$$u(x, y, t, z) = \sqrt{6} \tan\left(\frac{3\sqrt{6}}{2}\left(\frac{1}{3}(-x + y) + t^3\right) + \frac{3\sqrt{6}}{2}c_1\right) - \sqrt{6} \tan^{-1}\left(\tan\left(\frac{3\sqrt{6}}{2}\left(\frac{1}{3}(-x + y) + t^3\right) + \frac{3\sqrt{6}}{2}c_1\right)\right) \tag{39}$$

$$+ 9\left(\frac{1}{3}(-x + y) + t^3\right) + c_2 + 0.5tz^2 + t + \frac{27}{10}t^5 + \frac{3}{2}t^2(-x + y)$$

This solution is plotted in Fig. 5 for different values of times. It shows a change in the peak wave position as time increases and by increasing the time the surface of the wave follows a parabolic path whose amplitude decreases with time.

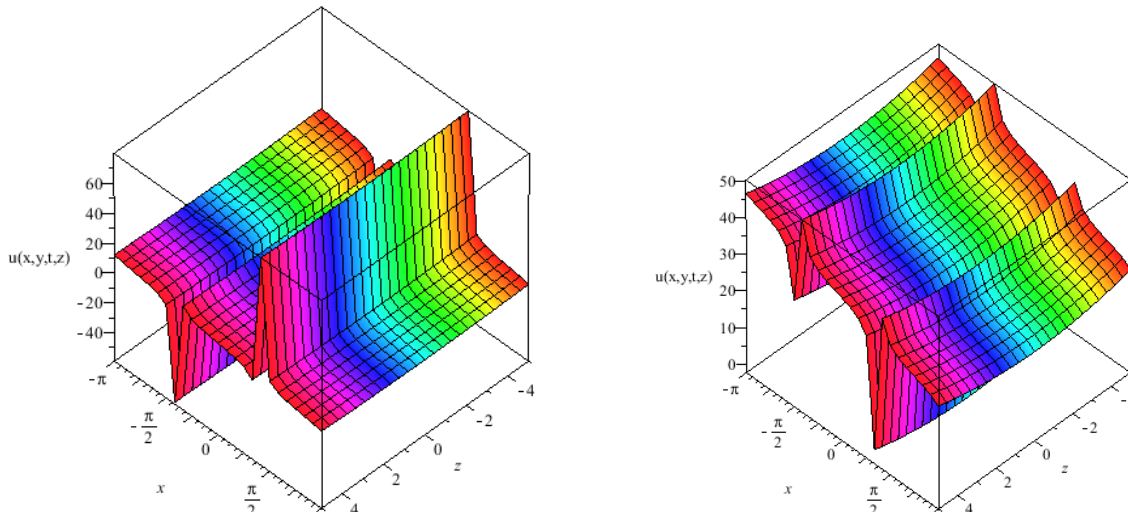


Figure 5. (a) $u(x, y, t, z)$ at $c_1 = 0, c_2 = 1, y = 0, t = 0.1$ sec.

(b) $u(x, y, t, z)$ at $c_1 = 0, c_2 = 1, y = 0, t = 0.3$ sec

Using $X_1 + X_9 + X_{13}$ Lie Vector to reduce the Independent Variables in (BKP) Equation

Equation (29) is transformed through the vector $X_1 + X_9 + X_{13} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + tz \frac{\partial}{\partial u}$ to;

$$K_{lh} + K_{lll} + 3K_{oll} + 3K_{ooll} + (K_{oooll}) - 6K_lK_{ll} - 9K_{lo}K_l - 3K_lK_{oo} + 3hK_l - 3K_{ll}K_o - 3K_{lo}K_o + 12K_{lo} + 3ohK_{ll} + 3ohK_{lo} + 6K_{ll} + 3K_{oo} - 3h = 0 \tag{40}$$

This equation has no closed-form solution, but possesses 12 Lie vectors; we choose V_1 ;

$$V_1 = 2 \frac{\partial}{\partial l} + \frac{\partial}{\partial o} + ho \frac{\partial}{\partial K} \tag{41}$$

Equation (40) is transformed to;

$$0.5s - \frac{3}{4}srF_{ss} - \frac{1}{16}F_{ssss} - 0.5F_{rs} - \frac{3}{4}rF_s + \frac{3}{4}F_sF_{ss} - \frac{3}{2}F_{ss} + \frac{3}{4}sr^2 - \frac{3}{2}r = 0 \tag{42}$$

This equation has no closed-form solution, but has eight Lie vectors; we choose to work only with e_5 ;

$$e_5 = \frac{\partial}{\partial r} + \frac{3}{2}r^2 \frac{\partial}{\partial s} + (1 + 0.5s^2 + \frac{9}{2}r^3s - 6rs) \frac{\partial}{\partial F} \tag{43}$$

As this Lie vector leads to an ODE with no analytic solution. Using e_5 , transform (BKP) to a nonlinear fourth degree ODE of the form;

$$-24\theta_{\eta\eta} - \theta_{\eta\eta\eta\eta} + 12\theta_{\eta\eta}\theta_{\eta} = 0 \tag{44}$$

Using Integrating Factor to get an explicit solution

We first deduce equation (44) IF using maple.

$$\mu_1 = \theta_\eta, \mu_2 = 1 \tag{45}$$

The integrating factor μ_1 reduces equation (44) to;

$$\theta_{\eta\eta}^2 - 4\theta_\eta^3 + 24\theta_\eta^2 = 0 \tag{46}$$

This equation has a closed-form solution;

$$\theta(\eta) = \sqrt{6} \tan(\sqrt{6}\eta + \sqrt{6}c_1) - \sqrt{6}\tan^{-1}(\tan(\sqrt{6}\eta + \sqrt{6}c_1)) + 6\eta + c_2 \tag{47}$$

where $\eta = -\frac{3}{2}r^3 + s$, $\theta(\eta) = F(r, s) - s + \frac{27}{10}r^5 + 3r^2s - (1 + \frac{1}{2}s^2)r$.

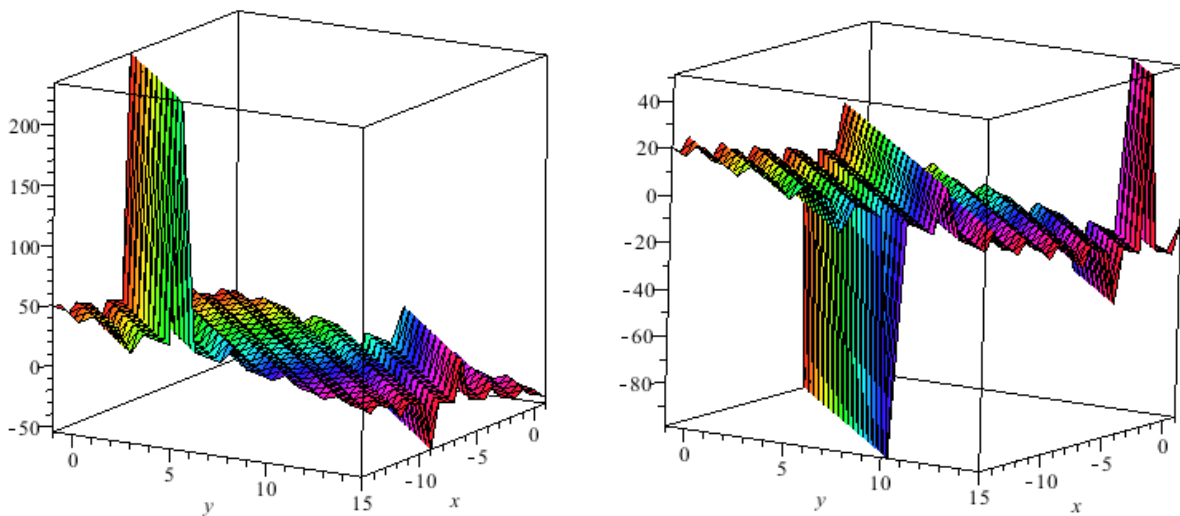
Then back to $(l; h; o)$ coordinates with

$$r = h, s = \frac{-l}{2} + o, F(r, s) = K(l, h, o) + \frac{1}{8}hl^2 - \frac{hlo}{2}; h = t, \\ l = -x + y, o = -x + z, K(l, h, o) = u(x, y, t, z) + 0.5x^2t - xtz$$

That leads to;

$$u(x, y, t, z) = \sqrt{6} \tan\left(\sqrt{6}\left(-\frac{3}{2}t^3 - \frac{x+y}{2} + z\right) + \sqrt{6}c_1\right) - \sqrt{6}\tan^{-1}\left(\tan\left(\sqrt{6}\left(-\frac{3}{2}t^3 - \frac{x+y}{2} + z\right) + \sqrt{6}c_1\right)\right) \\ + \left(1 + 0.5\left(\frac{-x-y}{2} + z\right)^2\right) + c_2 - 9t^3 - 3x - 3y + 6z + \frac{27}{10}t^5 - 3t^2\left(\frac{-x-y}{2} + z\right) \\ - 0.5x^2t + xtz - \frac{t}{8}(-x+y)^2 + \frac{t(-x+y)(-x+z)}{2} + \left(1 + \frac{1}{2}\left(\frac{-x-y}{2} + z\right)^2\right) \tag{48}$$

This wave is plotted in Fig. 6. A series of solitons inverse its flow at $t = 0.1$ sec.



(a)

(b)

Figure 6. (a) $u(x,y,t,z)$ at $c_1 = 1, c_2 = 1, z = 0.1$ at time $t = 0$; (b) $u(x,y,t,z)$ at $c_1 = 1, c_2 = 1, z = 0.1$ at time $t = 1$ sec.

The groups of soliton waves are decay with time and run towards left. Here, we compare our result in (48) with two solutions in (Krstic, 2009; Ren et al., 2013), we find that;

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- a) [Simsen & Simsen \(2011\)](#), used some of the generators and obtained some traveling wave solution. His solutions in most cases, localized only in three dependent variables (x, y, t) . We plot his result presented by X_4 in Fig. 7;

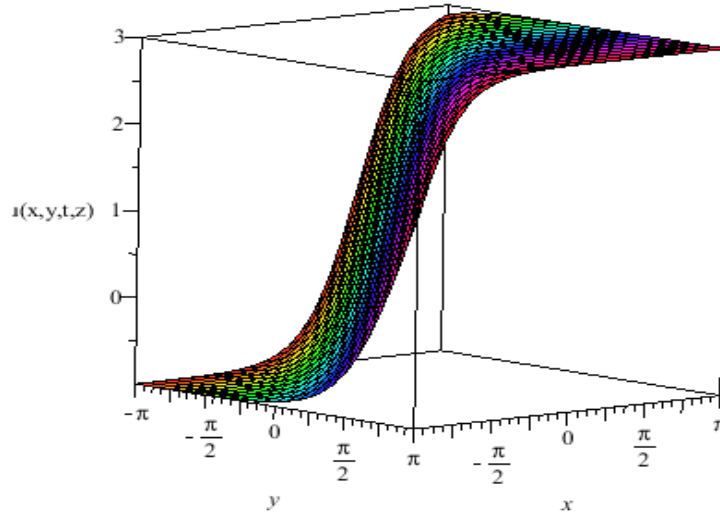


Figure 7. $u(x, y, t, z) = 2c_2 \tanh\left(\frac{1}{4}\left(\frac{4c_2c_2x+4c_2c_4t+4c_1c_2+3y}{c_2}\right)\right) + c_5$
for $c_1 = 1, c_2 = c_4 = c_5 =$ at time $t = 0$

- b) In Fig 7, he presented the travelling wave solution while we found a multi peaks solution as depicted in Fig. 6.

The solution in equation (23) presented in [Zgliczynski \(2003\)](#), in three variables (x, y, t) is different from our solution.

4. Conclusion

Here, we reduce the PDEs to ODEs through Lie vectors as previously done in [Bridges & Reich \(2001\)](#), through two reduction stages. Some of these ODEs have no solution. Some researchers in this step, use the SMM, power series method or Riccati equation method to solve non-solvable equations. We use the integrating factors as a tool to reduce the order and the nonlinearity in an ODE. This explores new solutions as it appears for the (2+1)-dimensional (CBS) and (3+1)-dimensional generalized BKP solutions compared with [Arcak \(2011\)](#) results.

Conflict of interest statement

The authors declared that they have no competing interest.

Statement of authorship

The authors have a responsibility for the conception and design of the study. The authors have approved the final article.

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